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Existence of pure strategy equilibrium in Bertrand-Edgeworth games with imperfect divisibility of money

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Abstract

This paper incorporates imperfect divisibility of money in a price game where a given number of identical firms produce a homogeneous product at constant unit cost up to capacity. We find necessary and sufficient conditions for the existence of a pure strategy equilibrium. Unlike in the continuous action space case, with discrete pricing there may be a range of symmetric pure strategy equilibria - which we fully characterize - a range which may or may not include the competitive price. Also, we determine the maximum number of such equilibria when competitive pricing is itself an equilibrium.

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1 Introduction

In theoretical work on strategic price setting in a homogeneous product industry, the price is customarily viewed as a continuous choice variable. This is an analytical simplification since there is in fact a minimum currency denomination (e. g., a cent). Interesting results have been achieved by incorporating discrete pricing in models of price competition. Consider first a setting where the firm output is demand determined. If the firms produce at the same constant marginal cost, then, besides the Bertrand solution (where the firms play a weakly dominated strategy), it is also an equilibrium for the firms to charge the lowest feasible price above marginal cost (Harrington, 1989). Discrete pricing has also been incorporated in models where average total cost is decreasing (due to a fixed and avoidable cost). Two results have been provided for the duopoly as the minimum fraction of the money unit converges to zero. In a symmetric duopoly, the equilibrium of the price game converges to the contestable outcome - average-cost pricing by a single producer (Chaudhuri, 1996). If instead one firm has an absolute cost advantage, then limit-pricing emerges at a perfect equilibrium (Chowdhury, 2002).

Discrete pricing is also relevant for the existence of pure strategy equilibrium (henceforth, PSE) in Bertrand-Edgeworth games. This line of research has been pursued by Dixon (1993) and more recently by Chowdhury (2008) under strict convexity of costs, a setting where a PSE does not exist in the continuous-action space version of the game (Tirole, 1988, p. 214).[1] Assuming the firms to choose first price and next output (each firm producing the minimum between its competitive supply at the set price and its forthcoming demand), a sufficient condition is established by Dixon for the existence of PSE under efficient rationing, a condition that holds in a sufficiently large industry. Chowdhury (2008) focuses mainly on a simultaneous price and quantity game. For a large class of rationing rules, he proves that, with sufficiently many identical firms, all firms charging the lowest feasible price above the competitive price [2] is the unique symmetric PSE.

With continuous prices, existence of PSE is also problematic when unit cost is constant up to capacity (Vives, 1986). We incorporate discrete pricing in this setup, assuming symmetric oligopoly, efficient rationing, and a decreasing demand meeting a concavity condition. Section 2 finds necessary

[1] On the other hand, a PSE would always exist if the firm met the whole of its forthcoming demand (Dastidar, 1995).

[2] Competitive price is identified with marginal cost at zero output, $c'(0)$. This identification relies on the following argument. Let there be $n < \infty$ price-taking potential entrants. Under strict cost convexity, there will be n active firms, and the competitive price will converge to $c'(0)$ (under perfect divisibility of money) as n increases.

and sufficient conditions for existence of a PSE and multiplicity of symmetric PSEs. The findings can be summarized as follows. A pivotal role is played by the highest uniform price that is not worth undercutting (p^*): this is at least as high as the competitive price (p^w). If competitive pricing is an equilibrium, then there are multiple symmetric PSEs so long as $p^* > p^w$, any uniform price from p^w to p^* being a PSE. If instead competitive pricing is not an equilibrium, then p^* is definitely higher than p^w and a PSE will exist if and only if a unilateral price increase is unworthy at market price p^*: then the set of symmetric PSEs will include any uniform price from p^* down to the lowest price at which a unilateral price increase is not worth it. Section 3 clarifies the role of the size of the market and the minimum currency denomination. As to the former, we derive - from an industry where a PSE does not exist - a family of larger industries by applying Dixon's (1993) "replication" procedure. In a sufficiently large replica industry, all firms charging p^* becomes a PSE and, with further increases in the industry size, the range of symmetric PSEs extends downwards to include competitive pricing. As to the minimum currency denomination (ϵ), if competitive pricing is not an equilibrium with continuous pricing, then no PSE exists with discrete pricing either, provided ϵ is sufficiently small. A second result relates to the case where there are several symmetric PSEs including competitive pricing. As ϵ decreases, the number of symmetric PSEs converges to a well-defined maximum while the price converges to p^w at any equilibrium. Section 4 briefly concludes and the Appendix contains most of the proofs.

2 The model

Each of n firms produces a homogeneous good at constant unit cost (normalized to zero) up to capacity $\overline{q} = \overline{Q}/n$, where $\overline{Q}$ is total capacity. $\mathcal{R}^+$ and $\mathcal{I}^+ = \{k\}$ are the sets of non-negative reals and integers, respectively. Demand and the inverse demand function are $D(p)$ and $P(Q)$, respectively, Q being total output. In the real domain, we take $D(p)$ to be twice continuously differentiable with $D(p) = 0$ for $p \geq \overline{\overline{p}}$, and $D(p) > 0$, $D'(p) < 0$ and $pD(p)$ weakly concave for $p \in (0, \overline{\overline{p}})$. We denote by $D'(p^\circ)$ the derivative of D at some specified price p°. The set of feasible prices is $\{k\epsilon\}$, $\epsilon > 0$ being the minimum currency denomination. The competitive price (p^w) is 0 if $D(0) \leq \overline{Q}$ and $P(\overline{Q})$ if $D(p) > \overline{Q}$ for p small enough. (In the latter case, we let $P(\overline{Q}) \in \{k\epsilon\}$, so that a competitive equilibrium exists.) In the price game the firms choose prices, whereupon the buyers make purchasing decisions. There are no costs to buyer mobility, hence a higher priced firm receives no residual demand unless the capacity of lower priced firms is fully utilized. $\pi_i(p_i, p_{-i})$ denotes firm i's payoff at strategy profile $(p_1, ..., p_n)$ and $\pi_i(p_i', p_{-i})$ its payoff if deviating to p_i'. Symmetric pure strategy profiles $(p, ..., p)$ are referred to as $\mathbf{p}$ and i's associated payoff as $\pi_i(\mathbf{p})$. Demand is equally split among equally priced firms, hence $\pi_i(\mathbf{p}) = pD(p)/n$ for $p > p^w$. Note that, with $p^w > 0$, $\pi_i(\mathbf{p}) = p\overline{q}$ at $p \in [0, p^w]$. We let $\overline{p} = \max\{p_1, ..., p_n\}$, $\overline{H} = \{i : p_i = \overline{p}\}$, and $\#\overline{H} = \overline{n}$. Rationing is according to the efficient rule: thus $\pi_i(p_i, p_{-i}) = p_i \min\{\overline{q}, \max\{0, \frac{D(p_i) - \#\{j: p_j < p_i\}\overline{q}}{1 + \#\{s \neq i : p_s = p_i\}}\}\}$ and, for $i \in \overline{H}$, $\pi_i(p_i, p_{-i}) = \overline{p} \min\{\overline{q}, \max\{0, \frac{D(\overline{p}) - (n - \overline{n})\overline{q}}{\overline{n}}\}\}$. Note that $p[D(p) - (n-1)\overline{q}] < \pi_i(\mathbf{p})$ for $p > p^w$. With $P((n-1)\overline{q}) > 0$ and $pD(p)$ *strictly* concave, we let $\widetilde{p}^\circ = \arg\max_{p \in \mathcal{R}^+} p[D(p) - (n-1)\overline{q}]$ and $\widetilde{\pi}^\circ = \widetilde{p}^\circ[D(\widetilde{p}^\circ) - (n-1)\overline{q}]$: in fact, then $\widetilde{p}^\circ$ is the unique solution to $d[p(D(p) - (n-1)\overline{q})]/dp = 0$.[3] In contrast, if $pD(p)$ is linear - i. e., $D(p) = (\alpha/p) - \beta$ (with $\alpha, \beta > 0$) for $p \leq \alpha/\beta$ - then $d[p(D(p) - (n-1)\overline{q})]/dp = -\beta$ for any $p > 0$. Further, we let $\widetilde{p} = \arg\max_{p \in \{k\epsilon\}} p[D(p) - (n-1)\overline{q}]$ and $\widetilde{\pi} = \widetilde{p}[D(\widetilde{p}) - (n-1)\overline{q}]$. Obviously, $\widetilde{p}$ is within $\pm\epsilon$ of $\widetilde{p}^\circ$.

It is easily seen when competitive pricing is an equilibrium.

Proposition 1 *(i)* $\mathbf{p^w}$ *is an equilibrium iff*

$$\frac{(p^w + \epsilon)\left[\overline{Q} - D(p^w + \epsilon)\right]}{\epsilon\overline{q}} \geq 1, \tag{1}$$

which can be written

$$(n-1)\overline{q} \geq D(\epsilon) \text{ if } p^w = 0. \tag{2}$$

(ii) In the continuous action space case ($\epsilon = 0$), $\mathbf{p^w}$ *is an equilibrium iff*

$$\frac{-p^w D'(p^w)}{\overline{q}} \geq 1 \text{ if } p^w > 0, \tag{1'}$$

and

$$(n-1)\overline{q} \geq D(0) \text{ if } p^w = 0. \tag{2'}$$

Proof. (i) (1) derives from $p^w\overline{q} \geq (p^w + \epsilon)[D(p^w + \epsilon) - (n-1)\overline{q}]$, which is necessary and also sufficient, by concavity of $pD(p)$. (ii) Similarly, (1') derives from $\frac{d}{dp}[p(D(p) - (n-1)\overline{q})]_{p = p^{w(+)}} \leq 0$; (2') is obvious. ■

Remark 1. (i) (1′) and (2′) are slightly stricter than (1) and (2), respectively: thus, if $\mathbf{p^w}$ is an equilibrium with $\epsilon = 0$, *a fortiori* it is so with $\epsilon > 0$. (ii) One can say that, with $\epsilon > 0$, $\mathbf{p^w}$ is not an equilibrium if and only if $\widetilde{\pi} > \pi_i(\mathbf{p^w})$, which in turn requires that $\widetilde{\pi}^\circ > \pi_i(\mathbf{p^w})$ (or, equivalently,

[3] $D(p) - (n-1)\overline{q} + pD'(p)$ is positive at $p = 0$ and negative for p sufficiently large.

$\widetilde{p}^{\circ} > p^w$). (iii) It can easily be checked that if $pD(p)$ is linear, then $p^w > 0$ and $\mathbf{p^w}$ is an equilibrium ((1') holds).

With $\epsilon = 0$, no $\mathbf{p} > \mathbf{p^w}$ can be an equilibrium: by infinitesimally undercutting, the firm's output jumps up and profit increases since the fall in revenue per unit is negligible. With discrete pricing, for any $\mathbf{p} > \mathbf{p^w}$ let $\Delta\Pi_{i|\Delta p_i}(\mathbf{p})$ be the change in i's profit if deviating from p by Δp_i. For $p \in [P(\overline{q}) + \epsilon, \overline{\overline{p}}]$, $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}) = (p-\epsilon)D(p-\epsilon) - (pD(p)/n)$, which is positive at $p > \epsilon$. Also, a unilateral price increase leads to zero profit when $D(p + \Delta p_i) \leq (n-1)\overline{q}$. Here are further results on $\Delta\Pi_{i|\Delta p_i}(\mathbf{p})$.

Lemma 1. *(i) With* $p \in (p^w, P(\overline{q}) + \epsilon]$, $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p})$ *is increasing in* p. *(ii) If* $\Delta\Pi_{i|\Delta p_i=\epsilon}(\mathbf{p^w}) \leq 0$, *then* $\Delta\Pi_{i|\Delta p_i}(\mathbf{p}) < 0$ *for any* $\Delta p_i > 0$ *and* $p \in (p^w, \overline{\overline{p}})$.

Proof. (i) $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}) = (p-\epsilon)\overline{q} - [pD(p)/n]$, which is increasing in p.

(ii) The statement is obvious if $p^w = 0$ since then $(n-1)\overline{q} \geq D(\epsilon)$ in the stated circumstances. If $p^w > 0$, then $\frac{d}{dp}[p(D(p) - (n-1)\overline{q})]_{p=p^w+\epsilon} < 0$ in the stated circumstances. Thus the statement follows straightforwardly, by concavity of $pD(p)$ and since $D(p) - (n-1)\overline{q} < D(p)/n$ at any $p > p^w$. ■

One necessary condition for $\mathbf{p} > \mathbf{p^w}$ to be an equilibrium is that at $\mathbf{p}$ it does not pay to undercut. $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}) \leq 0$ iff $(p-\epsilon)\overline{q} \leq pD(p)/n$, i. e., iff

$$p \leq \frac{\epsilon\overline{Q}}{\overline{Q} - D(p)}. \tag{3}$$

Let p^* be the highest $p \in \{k\epsilon\} \cap [p^w, P(\overline{q})]$ meeting (3) and let $\widehat{p}^* \in \mathcal{R}^+$ solve (3) as equality over the range $[p^w, P(\overline{q})]$. Of course, $\widehat{p}^* > p^w$ and $p^* = \{k\epsilon\} \cap (\widehat{p}^* - \epsilon, \widehat{p}^*]$. We have the following result on p^*.

Lemma 2. $p^* \geq \epsilon$ *when* $p^w = 0$, *with* $p^* \geq 2\epsilon$ *iff* $\overline{Q} \leq 2D(2\epsilon)$; $p^* \geq P(\overline{Q})$ *when* $p^w = P(\overline{Q}) > 0$, *with* $p^* \geq p^w + \epsilon$ *iff* $\frac{\overline{Q}-D(p^w+\epsilon)}{D(p^w+\epsilon)}\frac{p^w}{\epsilon} \leq 1$.

Proof. With $p^w = 0$, $p^* \geq 2\epsilon$ if $2 \leq \frac{\overline{Q}}{\overline{Q}-D(2\epsilon)}$, i. e., $\overline{Q} \leq 2D(2\epsilon)$, while $p^* = \epsilon$ if $\overline{Q} > 2D(2\epsilon)$. With $p^w > 0$, $p^* \geq p^w + \epsilon$ if (3) holds at $p = p^w + \epsilon$, i. e., $\frac{\overline{Q}-D(p^w+\epsilon)}{D(p^w+\epsilon)}\frac{p^w}{\epsilon} \leq 1$; if not, then $p^* = p^w$. ■

We can now address equilibrium multiplicity when $\mathbf{p^w}$ is an equilibrium.

Proposition 2 *Let* $\mathbf{p^w}$ *be an equilibrium. Then: (i) the set of symmetric PSEs is made up of any* $\mathbf{p} : p \in \{k\epsilon\} \cap [p^w, p^*]$; *(ii) if* $p^w = 0$ *there are further symmetric PSEs besides* $\mathbf{0}$ *and* $\boldsymbol{\epsilon}$ *iff* $\overline{Q} \leq 2D(2\epsilon)$; *if* $p^w > 0$ *there are further symmetric PSEs besides* $\mathbf{p^w}$ *iff* $\frac{\overline{Q}-D(p^w+\epsilon)}{D(p^w+\epsilon)}\frac{p^w}{\epsilon} \leq 1$.

Proof. (i) Since $\Delta\Pi_{i|\Delta p_i>0}(\mathbf{p^w}) \leq 0$, by Lemma 1 any price increase is unworthy for any $p > p^w$. Undercutting is also unworthy for any $p \in [p^w, p^*]$.

(ii) This follows from Lemma 2. ■

Examples. 1: $D(p) = 50.4 - 12p$, $n = 24$, $\overline{q} = 2, \epsilon = .01$. Then $p^w = .20$, and $\mathbf{p^w}$ is an equilibrium ((1) holds); $p^* = .32$, hence any $\mathbf{p} : p \in \{.20, .21, ..., .32\}$ is a PSE. 2: $D(p) = 60 - 10p$, $n = 10$, $\overline{q} = 2$, $\epsilon = .01$. Then $p^w = 4$; $\mathbf{p^w}$ is an equilibrium and a unique one since $p^* = 4$.

A PSE may exist even when $\mathbf{p^w}$ is not an equilibrium. Before seeing this, two points must preliminarily be made. First we have

Lemma 3. *With $p < \overline{\overline{p}}$ and $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon}) \geq 0$, $\Delta\Pi_{i|\Delta p_i=\epsilon}(\mathbf{p}) < 0$.*

Proof. In the Appendix. ■

Secondly, when $\mathbf{p^w}$ is not an equilibrium, undercutting is definitely unworthy for $\mathbf{p}$ close enough to $\mathbf{p^w}$.

Lemma 4. *Suppose $\mathbf{p^w}$ is not an equilibrium. Then: (i) $p^* \in [p^w + \epsilon, P(\overline{q})]$; (ii) $p^* \in [2\epsilon, P(\overline{q})]$ if $p^w = 0$ and $\boldsymbol{\epsilon}$ is not an equilibrium; (iii) $\pi_i(\mathbf{p})$ is increasing for $p \in \{p^w, p^w + \epsilon, ..., p^*\}$.*

Proof. In the Appendix. ■

Now, $\mathbf{p}^*$ is an obvious equilibrium candidate since at $\mathbf{p}^*$ an ϵ-price increase is unworthy (by Lemma 3, $\Delta\Pi_{i|\Delta p_i=\epsilon}(\mathbf{p}^*) < 0$ because $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}^*+\boldsymbol{\epsilon}) > 0$). However, for $\mathbf{p}^*$ to be an equilibrium it has to be $\Delta\Pi_{i|\Delta p_i}(\mathbf{p}^*) \leq 0$ for *any* $\Delta p_i > 0$. We have this result.

Proposition 3 *Let $\mathbf{p^w}$ not be an equilibrium. Then: (i) $\mathbf{p}^*$ is an equilibrium iff $\pi_i(\mathbf{p}^*) \geq \widetilde{\pi}$. Holding this, let $p^{**} \in \{k\epsilon\} \cap (p^w, p^*]$ be such that $\pi_i(\mathbf{p}^{**}-\boldsymbol{\epsilon}) < \widetilde{\pi} \leq \pi_i(\mathbf{p}^{**})$. The set of symmetric PSEs is made up of any $\mathbf{p} : p \in \{k\epsilon\} \cap [p^{**}, p^*]$. (ii) There are no (symmetric or asymmetric) PSEs iff $\widetilde{\pi} > \pi_i(\mathbf{p}^*)$.*

Proof. In the Appendix. ■

Remark 2. (i) By concavity of $pD(p)$ and since $p[D(p) - (n-1)\overline{q}] < \pi_i(\mathbf{p})$ for $p > p^w$, a *sufficient* condition for $\mathbf{p}^*$ to be an equilibrium is $\widetilde{p} \leq p^*$. (ii) If $\mathbf{p^w}$ is not an equilibrium, there will normally be several (if any) symmetric PSEs, $\mathbf{p}^*$ being the only one iff $\pi_i(\mathbf{p}^*-\boldsymbol{\epsilon}) < \widetilde{\pi} \leq \pi_i(\mathbf{p}^*)$.

3 Comparative statics

We now see how the size of the market and of the minimum currency denomination affect the equilibrium. Let us begin with the former. Given ϵ and $\overline{q}$, an industry is a "demand function-number of firms" pair. Suppose there is no PSE in industry $(D(p), n)$: by Lemma 4 and Prop. 3, $\widetilde{p} > p^* > p^w$ and $\widetilde{\pi} > \pi_i(\mathbf{p}^*) > \pi_i(\mathbf{p^w})$. To generate industries of different size, we adopt Dixon's (1993) replication procedure: from $(D(p), n)$ a family of industries is derived, $(D^{(r)}(p), n^{(r)})$ $=(rD(p), rn)$, where $r > 1$ and $rn \in \mathcal{I}^+$. Letting x be the value of some variable in industry $(D(p), n)$, $x(r)$ denotes its value in the "r-replica" industry. Note that $p^w(r) = p^w$, $p^*(r) = p^*$, and $\pi_i(\mathbf{p}(r)) = \pi_i(\mathbf{p})$ while $\widetilde{p}^{\circ}(r)$ is decreasing in r and $\widetilde{\pi}^{\circ}(r)$ is also decreasing (so long as $\widetilde{p}^{\circ}(r) > p^w$): for $D(0) \neq \overline{Q}$, $\widetilde{p}^{\circ}(r) = p^w$ and $\widetilde{\pi}^{\circ}(r) = \pi_i(\mathbf{p^w})$ at some $r \in \mathcal{R}^+$. (With $D(0) = \overline{Q}$, $\widetilde{p}^{\circ}(r)$ converges asymptotically to 0.) It follows immediately that $\mathbf{p}^*$ is an equilibrium in a sufficiently large r-replica industry; also, the set of symmetric PSEs includes $\mathbf{p^w}$ for r large enough.

Proposition 4 *Let there be no PSE in industry* $(D(p), n)$. *Then, in industry* $(D^{(r)}(p), n^{(r)})$: *(i)* $\mathbf{p}^*$ *is an equilibrium for any* $r \geq r'$, r' *being the smallest* r *such that* $rn \in \mathcal{I}^+$ *and* $\widetilde{\pi}(r) \leq \pi_i(\mathbf{p}^*)$; *(ii) with* $r \geq r'$, *the set of symmetric PSEs is made up of any* $\mathbf{p} : p \in \{k\epsilon\} \cap [p^{**}(r), p^*]\}$ *where* $p^{**}(r)$ *is non-increasing in* r *and* $p^{**}(r) = p^w$ *for* $r \geq r''$, r'' *being the smallest* r *such that* $rn \in \mathcal{I}^+$ *and* $(p^w + \epsilon)[rD(p^w + \epsilon) - (rn - 1)\overline{q}] \leq \pi_i(\mathbf{p^w})$ *(i. e.,* $r(p^w + \epsilon)\left[\overline{Q} - D(p^w + \epsilon)\right] \geq \epsilon\overline{q}$ *if* $p^w > 0$ *and* $r[\overline{Q} - D(\epsilon)] \geq \overline{q}$ *if* $p^w = 0$*).*[4]

Example. With $\epsilon = .01$ and $\overline{q} = 2$, let $(D(p) = 4.2 - p, n = 2)$. Then $p^w = .2$, $\pi_i(\mathbf{p^w}) = .4$, $p^* = .32$, and $\pi_i(\mathbf{p}^*) = .62$. There is no PSE: $\widetilde{p} = 1.1$ and $\widetilde{\pi} = 1.21 > \pi_i(\mathbf{p}^*) > \pi_i(\mathbf{p^w})$. One can check that $r' = 3$, hence $\mathbf{p}^*$ is an equilibrium in any r-replica industry with $r \geq 3$. In fact, $\widetilde{p}(r') = .43$ and $\widetilde{\pi}(r') = .56 < .62$. Further, $p^{**}(r') = .29$, so that there are four symmetric PSEs in the r'-replica industry. As r increases the set of symmetric PSEs increases: with $r \geq r'' = 10$ any $p \in \{p^w, p^w + \epsilon, ..., p^*\}$ is an equilibrium.

To see the relevance of the size of the minimum currency denomination, we now allow for changes in ϵ (adjusting notation accordingly) while taking $D(p), n,$ and $\overline{q}$ as given. By the way, with $P(\overline{Q}) > 0$ only values of ϵ such that $P(\overline{Q}) \in \{k\epsilon\}$ can be allowed for: thus $P(\overline{Q})$ is the maximum feasible value for ϵ. In the real domain, $\widehat{p}^*(\epsilon)$ is continuous and twice differentiable with $d\widehat{p}^*(\epsilon)/d\epsilon > 0$ and $\lim_{\epsilon \to 0} \widehat{p}^*(\epsilon) = p^w$. As to $p^*(\epsilon)$ and $\pi_i(\mathbf{p}^*(\epsilon))$, we have

Lemma 5. *(i) Let* $\overline{Q} \geq 2D(0)$ *if* $p^w = 0$ *and* $\frac{-D'(p^w)}{\overline{Q}} p^w \geq 1$ *if* $p^w > 0$. *Then, for any* $\epsilon > 0$, $p^*(\epsilon) = \epsilon$ *if* $p^w = 0$ *and* $p^*(\epsilon) = p^w$ *if* $p^w > 0$. *(ii) Let* $\overline{Q} < 2D(0)$ *if* $p^w = 0$ *and* $\frac{-D'(p^w)}{\overline{Q}} p^w < 1$ *if* $p^w > 0$. *Then, for* ϵ *small enough: (ii.a)* $p^*(\epsilon) \geq 2\epsilon$ *if* $p^w = 0$ *and* $p^*(\epsilon) \geq p^w + \epsilon$ *if* $p^w > 0$; *(ii.b)* $p^*(\epsilon)$ *is non-decreasing in* ϵ, $\lim_{\epsilon \to 0} p^*(\epsilon) = p^w$ *and* $\lim_{\epsilon \to 0} \pi_i(\mathbf{p}^*(\epsilon)) = \pi_i(\mathbf{p^w})$.

[4] For comparison, let us review the main point made by Dixon through his replication procedure under strict cost convexity. As in our model, $p^w(r) = p^w$ and $p^*(r) = p^*$. Now, at $\mathbf{p}^*$ let firm i deviate to $p^* + \epsilon$: then its residual demand is decreasing in r and falls to zero when r increases above some critical level (call it $\widehat{r}$): $r > \widehat{r}$ is the condition Dixon draws attention to - clearly, a *sufficient* condition for $\mathbf{p}^*$ to be an equilibrium.

Proof. (i), (ii.a) These follow from Lemma 2, $D' < 0$ (as far as case $p^w = 0$ is concerned) and concavity of $pD(p)$ (as far as case $p^w > 0$ is concerned).

(ii.b) This follows from $d\widehat{p}^*(\epsilon)/d\epsilon > 0$ and $\lim_{\epsilon\to 0}\widehat{p}^*(\epsilon) = p^w$. ■

This is a crucial result since $\mathbf{p} \leq \mathbf{p}^*$ at any symmetric PSE. One consequence is that, if $\mathbf{p^w}$ is not an equilibrium with continuous prices, then no PSE exists with discrete pricing provided ϵ is small enough.

Proposition 5 *Suppose $\mathbf{p^w}$ is not an equilibrium in the continuous action space case. Then, if $\epsilon > 0$ is sufficiently small, no PSE exists.*

Proof. In the stated conditions, $\pi_i(\mathbf{p^w}) < \widetilde{\pi}^\circ$. Note that $\lim_{\epsilon\to 0}\widetilde{p}(\epsilon) = \widetilde{p}^\circ$ and $\lim_{\epsilon\to 0}\widetilde{\pi}(\epsilon) = \widetilde{\pi}^\circ$. Thus, for ϵ sufficiently small, $\mathbf{p^w}$ is not an equilibrium: $\pi_i(\mathbf{p^w}) < \widetilde{\pi}(\epsilon)$. Furthermore, by Lemma 5, for ϵ sufficiently small $\pi_i(\mathbf{p}^*(\epsilon)) < \widetilde{\pi}(\epsilon)$ and no PSE exists. ■

The size of ϵ also matters when $\mathbf{p^w}$ is an equilibrium. Let $h^* \in \mathcal{I}^+ : p^*(\epsilon) = p^w + (h^* - 1)\epsilon$, so that $h^* = \#\{\{k\epsilon\} \cap [p^w, p^*(\epsilon)]\}$. Note that h^* is the number of symmetric PSEs when p^w is itself an equilibrium. We also let $\widehat{h}^* \in \mathcal{R}^+ : \widehat{p}^*(\epsilon) = p^w + (\widehat{h}^* - 1)\epsilon$, hence $\widehat{h}^* = [(\widehat{p}^*(\epsilon) - p^w)/\epsilon] + 1$ while $h^* = [(p^*(\epsilon) - p^w)/\epsilon] + 1$. We can now address equilibrium multiplicity in the event of $\mathbf{p^w}$ being an equilibrium.

Proposition 6 *Suppose $\mathbf{p^w}$ is an equilibrium in the continuous action space case. (i) Let $\overline{Q} \geq 2D(0)$ if $p^w = 0$ and $\frac{-D'(p^w)}{\overline{Q}}p^w \geq 1$ if $p^w > 0$. Then, for any $\epsilon > 0$: $\mathbf{0}$ and $\boldsymbol{\epsilon}$ are the only symmetric PSEs ($h^* = 2$) if $p^w = 0$ and $\mathbf{p^w}$ is the unique symmetric PSE ($h^* = 1$) if $p^w > 0$. (ii) Let $\overline{Q} < 2D(0)$ if $p^w = 0$ and $\frac{-D'(p^w)}{\overline{Q}}p^w < 1$ if $p^w > 0$. Then: (ii.a) with $\epsilon > 0$ small enough, $h^* \geq 3$ if $p^w = 0$ and $h^* \geq 2$ if $p^w > 0$; (ii.b) $dh^*/d\epsilon < 0$ at any $\epsilon > 0$ and $\max h^* = \lim_{\epsilon\to 0} h^* = \mathcal{I}^+ \cap [\frac{\overline{Q}}{\overline{Q} - D(p^w) - p^w D'(p^w)}, \frac{\overline{Q}}{\overline{Q} - D(p^w) - p^w D'(p^w)} + 1)$.*

Proof. In the Appendix. ■

Note that, when there are several symmetric PSEs including $\mathbf{p}^w$, any equilibrium price converges to p^w as $\epsilon \to 0$: this is because $\mathbf{p} \leq \mathbf{p}^*$ at any symmetric PSE and $\lim_{\epsilon\to 0} p^*(\epsilon) = p^w$.

Examples. Here are two examples to illustrate statement (ii) of Prop. 6.

1: $D(p) = 50.4 - 12p$, $n = 24$, $\overline{q} = 2$. Then $p^w = .2$ and $\mathbf{p}^w$ is an equilibrium for any ϵ. For $\epsilon = .2$, $p^*(\epsilon) = 1$ and $h^* = 5$; for $\epsilon = .01$, $p^*(\epsilon) = .32$ and $h^* = 13$; for $\epsilon = .0001$, $p^*(\epsilon) = 0.2019$ and $h^* = 20$. Note that $\max h^* = \mathcal{I}^+ \cap [20, 21) = 20$.

2: $D(p) = 52 - p$, $n = 14$, $\overline{q} = 2$. Then $p^w = 24$ and $\mathbf{p}^w$ is an equilibrium for any ϵ. For $\epsilon > 4$, $p^*(\epsilon) = 24$ and $h^* = 1$: $\mathbf{p}^w$ is the unique symmetric PSE. For any $\epsilon \leq 4$, $p^*(\epsilon) = p^w + \epsilon$ ($h^* = \max h^* = \mathcal{I}^+ \cap [\frac{28}{24}, \frac{52}{24}) = 2$).

4 Conclusion

We have studied discrete pricing when identical price-setting firms produce a homogeneous commodity at constant unit cost up to capacity. Necessary and sufficient conditions have been found for the existence of a PSE and for multiplicity of symmetric PSEs. We have seen that, with discrete pricing, there may exist a PSE even when competitive pricing is not an equilibrium, although such an event does not occur when the minimum fraction (ϵ) of the money unit is sufficiently small. Also, the existence of several symmetric PSEs including competitive pricing is a concrete possibility and we have computed the maximum number of such equilibria, obtaining for ϵ small enough.

Thus discrete pricing may lead to quite different results compared to the continuous-action space model. On the other hand, one basic prediction of that model - that the firms earn the competitive profit at any PSE of the price game - is not fundamentally misleading: if ϵ is sufficiently small, then either a PSE does not exist or the price must be equal to or cannot differ significantly from the competitive price at any symmetric PSE.

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Appendix

Proof of Lemma 3. Since $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon}) \geq 0$, then $p+\epsilon > p^w$ if $p^w > 0$ and $p+\epsilon \geq 2\epsilon$ if $p^w = 0$. Let $\Delta q_{i|\Delta p_i}(\mathbf{p})$ be the change in i's output when deviating from p by Δp_i: then $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon}) = p\Delta q_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon}) - \epsilon\frac{D(p+\epsilon)}{n}$ and $\Delta\Pi_{i_{|\Delta p_i=\epsilon}}(\mathbf{p}) = p\Delta q_{i|\Delta p_i=\epsilon}(\mathbf{p}) + \epsilon \max\{0, D(p+\epsilon)-(n-1)\overline{q}\}$. Our point is obvious if $p \geq P((n-1)\overline{q})-\epsilon$, hence we focus on $p \in (p^w, P((n-1)\overline{q})-\epsilon)$. Here, $-\Delta q_{i|\Delta p_i=\epsilon}(\mathbf{p}) = [D(p)/n]-[D(p+\epsilon)-(n-1)\overline{q}]$ and $\Delta q_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon}) = \overline{q} - [D(p+\epsilon)/n]$. Letting $-\Delta q_{i|\Delta p_i=\epsilon}(\mathbf{p}) = \Delta q_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon}) + \delta$ (where $\delta > 0$),[5] it is found that $-\Delta\Pi_{i_{|\Delta p_i=\epsilon}}(\mathbf{p}) = p[\Delta q_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon})+\delta] - \epsilon[D(p+\epsilon)-(n-1)\overline{q}]$. The right-hand side is larger than $\Delta\Pi_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon})$ since $D(p+\epsilon)-(n-1)\overline{q} < \frac{D(p+\epsilon)}{n}$. Thus $\Delta\Pi_{i_{|\Delta p_i=\epsilon}}(\mathbf{p}) < 0$. ■

Proof of Lemma 4. (i) Since $\Delta\Pi_{i_{|\Delta p_i=\epsilon}}(\mathbf{p^w}) > 0$, then by Lemma 3, $\Delta\Pi_{i_{|\Delta p_i=-\epsilon}}(\mathbf{p^w}+\boldsymbol{\epsilon}) < 0$: at $\mathbf{p^w}+\boldsymbol{\epsilon}$ it does not pay to undercut. Further, $p^* \leq P(\overline{q})$ since profit is certainly raised by undercutting at $p > P(\overline{q})$ and such that $D(p) > 0$.

(ii) The argument runs as above.

(iii) The statement follows from concavity of $\pi_i(\mathbf{p})$ and $\pi_i(\mathbf{p}^*) \geq (p^*-\epsilon)\overline{q} \geq \pi_i(\mathbf{p}^*-\boldsymbol{\epsilon})$, where at least one inequality is strict. This last fact is obvious when $p^* > p^w+\epsilon$ since then $(p^*-\epsilon)\overline{q} > \pi_i(\mathbf{p}^*-\boldsymbol{\epsilon}) = (p^*-\epsilon)D(p^*-\epsilon)/n$. It is also obvious when $p^* = p^w+\epsilon = \epsilon$: then $\pi_i(\mathbf{p}^*) > \pi_i(\mathbf{p}^*-\boldsymbol{\epsilon}) = 0$. When $p^* = p^w+\epsilon > \epsilon$, we distinguish among two cases. If $\mathbf{p}^*$ is an equilibrium, then $\pi_i(\mathbf{p}^*) \geq \widetilde{\pi}$ while $\widetilde{\pi} > \pi_i(\mathbf{p}^*-\boldsymbol{\epsilon})$ since $p^w = p^*-\epsilon$ and p^w is not an equilibrium. If p^* is not an equilibrium, then $\widetilde{\pi} > \pi_i(\mathbf{p}^*)$ and $\widetilde{p} > p^*$. If it were $\pi_i(\mathbf{p}^*) = \pi_i(\mathbf{p}^*-\boldsymbol{\epsilon})$ it would be $\left[\frac{d\pi_i(\mathbf{p})}{dp}\right]_{p=p^*} < 0$ and hence *a fortiori* $\left[\frac{d}{dp}(pD(p)-(n-1)\overline{q})\right]_{p=p^*} < 0$, contrary to the fact that $\widetilde{\pi} > \pi_i(\mathbf{p}^*)$. ■

Proof of Proposition 3. (i) If $\pi_i(\mathbf{p}^*) \geq \widetilde{\pi}$, a unilateral price increase is unworthy at $\mathbf{p}^*$. It is also unworthy at $\mathbf{p}^{**}$ and, by statement (iii) of Lemma 4, at any $\mathbf{p}$ in between. Undercutting is also unworthy, by definition of $\mathbf{p}^*$.

(ii) Given statement (iii) of Lemma 4, it follows from $\widetilde{\pi} > \pi_i(\mathbf{p}^*)$ that $\widetilde{\pi} > \pi_i(\mathbf{p})$ at $\mathbf{p} < \mathbf{p}^*$: at any $\mathbf{p} < \mathbf{p}^*$, i's profit is raised by deviating to

[5] Note that $-\Delta q_{i|\Delta p_i=\epsilon}(\mathbf{p}) > \Delta q_{i|\Delta p_i=-\epsilon}(\mathbf{p}+\boldsymbol{\epsilon})$ since $(n-1)[\overline{Q}-D(p+\epsilon)] > [\overline{Q}-D(p)]$.

$\widetilde{p}$. Next, we dispose of asymmetric strategy profiles with $D(\overline{p}) < \overline{Q}$ (those with $D(\overline{p}) \geq \overline{Q}$ are immediately ruled out.) Let $\pi_i(p_i, p_{-i}) > 0$ for $i \in \overline{H}$ (otherwise our case is obvious), so that $D(\overline{p}) - (n - \overline{n})\overline{q} > 0$ and $\pi_j(p_j, p_{-j}) = p_j\overline{q}$ for $j \notin \overline{H}$. If $p_j < \overline{p} - \epsilon$ for some j, then any such j has not made a best reply because $\pi_j(p'_j, p_{-j}) = p'_j\overline{q}$ for $p'_j \in (p_j, \overline{p})$.

We are left with strategy profiles such that $D(\overline{p}) < \overline{Q}$ and $p_j = \overline{p} - \epsilon$ for all $j \notin \overline{H}$. Suppose first $D(\overline{p} - \epsilon) - (n - \overline{n})\overline{q} \geq \overline{q}$. If i's profit (for $i \in \overline{H}$) is not raised by deviating to $\overline{p} - \epsilon$, i. e., $(\overline{p} - \epsilon)\overline{q} \leq \overline{p}\frac{D(\overline{p}) - (n-\overline{n})\overline{q}}{\overline{n}}$, then it pays for $j \notin \overline{H}$ to deviate to $\overline{p}$: it is $(\overline{p} - \epsilon)\overline{q} < \overline{p}\frac{D(\overline{p}) - (n-1-\overline{n})\overline{q}}{1+\overline{n}}$ because $\frac{D(\overline{p}) - (n-1-\overline{n})\overline{q}}{1+\overline{n}} > \frac{D(\overline{p}) - (n-\overline{n})\overline{q}}{\overline{n}}$. Note that it is necessarily $D(\overline{p} - \epsilon) - (n - \overline{n})\overline{q} > \overline{q}$ when $\overline{n} > 1$ and $\overline{p} \leq \widetilde{p}$, the last inequality implying $D(\overline{p}) - (n-1)\overline{q} > 0$. Next consider strategy profiles such that $\overline{p} > \widetilde{p}$, $\overline{n} > 1$, and $D(\overline{p} - \epsilon) - (n - \overline{n})\overline{q} < \overline{q}$. Then it pays for $i \in \overline{H}$ to deviate to $\overline{p} - 2\epsilon$, i. e., $(\overline{p} - 2\epsilon)\overline{q} > \overline{p}\frac{D(\overline{p}) - (n-\overline{n})\overline{q}}{\overline{n}}$. In fact, this condition amounts to $\frac{2\epsilon}{\overline{p}}\frac{\overline{n}\overline{q}}{\overline{Q} - D(\overline{p})} < 1$, which certainly holds: $\frac{\overline{n}\overline{q}}{\overline{Q} - D(\overline{p})} \in (1, 2)$ since $D(\overline{p}) < (n - \overline{n})\overline{q} + \overline{q}$ and $\overline{n} > 1$, and $2\epsilon/\overline{p} \leq 1/2$ since $\overline{p} \geq 4\epsilon$ (due to $\overline{p} > \widetilde{p} > p^* > \epsilon$). Finally, consider strategy profiles such that $\overline{n} = 1$. These are easily dismissed if $\overline{p} \neq \widetilde{p}$.[6] If instead $\overline{p} = \widetilde{p}$, then $i \in \overline{H}$ would be better off by deviating to $\overline{p} - 2\epsilon$. To see this, note that, since $\widetilde{p} > p^*$, at $\widetilde{\mathbf{p}}$ it pays to undercut, hence $\epsilon n\overline{q} < \widetilde{p}[\overline{Q} - D(\widetilde{p})]$. Consequently, at the asymmetric strategy profiles under consideration it pays $i \in \overline{H}$ to deviate to $\widetilde{p} - 2\epsilon$: the resulting payoff of $(\widetilde{p} - 2\epsilon)\overline{q}$ can in fact be checked to be higher than $\widetilde{p}[D(\widetilde{p}) - (n-1)\overline{q}]$ so long as $2\epsilon\overline{q} < \widetilde{p}[\overline{Q} - D(\widetilde{p})]$. ■

Proof of Proposition 6. (i) and (ii.a) follow from Lemma 2 and concavity of $pD(p)$. Incidentally, it is easily checked that if $pD(p)$ is linear, then $\mathbf{p^w}$ is the unique symmetric PSE ($-D'(p^w)p^w/\overline{Q} \geq 1$).

(ii.b) By the definition of $\widehat{h}^*$ and using l'Hopital's rule, $\lim_{\epsilon\to 0}\widehat{h}^* = \lim_{\epsilon\to 0}(d\widehat{p}^*(\epsilon)/dp) + 1$. Since $(\widehat{p}^* - \epsilon)\overline{q} - [\widehat{p}^*D(\widehat{p}^*)/n] = 0$, it is $d\widehat{p}^*/d\epsilon = \frac{\overline{Q}}{\overline{Q} - D(\widehat{p}^*) - \widehat{p}^*D'(\widehat{p}^*)}$ and hence $\lim_{\epsilon\to 0}\widehat{h}^* = \frac{\overline{Q}}{\overline{Q} - D(p^w) - p^wD'(p^w)} + 1$, which equals $\frac{\overline{Q}}{\overline{Q} - D(p^w)} + 1$ if $p^w = 0$ and $\frac{\overline{Q}}{-p^wD'(p^w)} + 1$ if $p^w > 0$.

Next we show that $d\widehat{h}^*/d\epsilon < 0$ at any $\epsilon > 0$ so that $\lim_{\epsilon\to 0}\widehat{h}^* = \max\widehat{h}^*$. By the definition of $\widehat{p}^*$ and $\widehat{h}^*$,

$$\widehat{h}^* - 1 - \frac{\overline{Q}}{\overline{Q} - D(p^w + (\widehat{h}^* - 1)\epsilon)} + \frac{p^w}{\epsilon} = 0. \tag{4}$$

With $p^w = 0$, it is immediately seen that $d\widehat{h}^*/d\epsilon < 0$. More generally, implicit

[6] In particular, with $\overline{p} > \widetilde{p}$, deviating to $\overline{p} - \epsilon$ yields $(\overline{p} - \epsilon)D(\overline{p} - \epsilon)/n$, higher than $(\overline{p} - \epsilon)[D(\overline{p} - \epsilon) - (n-1)\overline{q}]$, in its turn higher than i's initial payoff, $\overline{p}[D(\overline{p}) - (n-1)\overline{q}]$.

differentiation of (4) yields

$$\frac{d\widehat{h}^*}{d\epsilon} = \frac{\overline{Q}D'(\widehat{h}^* - 1)\epsilon^2 + (\overline{Q} - D)^2 p^w}{\epsilon^2[(\overline{Q} - D)^2 - \epsilon\overline{Q}D']}, \tag{5}$$

where D' and D are evaluated at $p = \widehat{p}^*$. For further use below, note that

$$\widehat{h}^* = \frac{\overline{Q}}{\overline{Q} - D(\widehat{p}^*) - \widehat{p}^* D'(\widehat{p}^*)} + 1 - \epsilon\frac{d\widehat{h}^*}{d\epsilon}. \tag{5'}$$

Making use of (4), from (5) it is easily seen that $d\widehat{h}^*/d\epsilon < 0$ if and only if $\overline{Q}\epsilon(\widehat{p}^* - p^w)\left(D' + \frac{\overline{Q}-D}{\widehat{p}^*-p^w}\right) - (\widehat{p}^* - p^w)(\overline{Q} - D)^2 < 0$. With $p^w > 0$ and $D'' \leq 0$, this inequality obviously holds for any ϵ, since then $\frac{\overline{Q}-D}{\widehat{p}^*-p^w} \leq -D'$.

With $p^w > 0$, $d\widehat{h}^*/d\epsilon < 0$ even if $D'' > 0$, by concavity of $pD(p)$. To see this, we begin by showing that $d\widehat{h}^*/d\epsilon < 0$ for ϵ close enough to 0. Suppose contrarise that, in a right neighbourhood of 0, $d\widehat{h}^*/d\epsilon \geq 0$ so that $\widehat{h}^* \geq \lim_{\epsilon\to 0} \widehat{h}^*$ in that neighbourhood. On the other hand, if $d\widehat{h}^*/d\epsilon \geq 0$, then from (5'), strict concavity [7] of $pD(p)$, and $d\widehat{p}^*(\epsilon)/d\epsilon > 0$ it follows that $\widehat{h}^* < \lim_{\epsilon\to 0} \widehat{h}^*$: a contradiction. Next we see that $d\widehat{h}^*/d\epsilon < 0$ throughout $(0, p^w]$. If not, then a local minimum occurs at some $\epsilon^{\circ\circ} \in (0, p^w)$, and there exists $\epsilon^\circ > \epsilon^{\circ\circ}$ such that $d\widehat{h}^*/d\epsilon \geq 0$ and $\widehat{h}^*(\epsilon^\circ) \geq \widehat{h}^*(\epsilon^{\circ\circ})$. On the other hand, $\widehat{p}^*(\epsilon^{\circ\circ}) < \widehat{p}^*(\epsilon^\circ)$; hence, by concavity of $pD(p)$ and since $d\widehat{h}^*/d\epsilon = 0$ at $\epsilon^{\circ\circ}$ and $d\widehat{h}^*/d\epsilon \geq 0$ at ϵ°, it follows from (5') that $\widehat{h}^*(\epsilon^{\circ\circ}) > \widehat{h}^*(\epsilon^\circ)$: a contradiction.

All the above has immediate implications for h^*, with $p^w > 0$ or $p^w = 0$. Since $h^* = \mathcal{I}^+ \cap (\widehat{h}^* - 1, \widehat{h}^*]$, h^* increases or stays constant as ϵ decreases: finally, $\lim_{\epsilon\to 0} h^*(\epsilon) = \mathcal{I}^+ \cap [\frac{\overline{Q}}{\overline{Q}-D(p^w)-p^w D'(p^w)}, \frac{\overline{Q}}{\overline{Q}-D(p^w)-p^w D'(p^w)} + 1)$ since, at any $\epsilon > 0$, $\widehat{h}^* < \lim_{\epsilon\to 0} \widehat{h}^*$. ■

[7] As already noted, with a linear $pD(p)$ we are necessarily in the circumstances of statement (i) ($\mathbf{p}^w$ is the unique symmetric PSE).